Network repair based on community structure

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Abstract – Real-world complex systems are often fragile under disruptions. Accordingly, research on network repair has been studied intensively. Recently proposed efficient strategies for network disruption, based on collective influence, call for more research on efficient network repair strategies. Existing strategies are often designed to repair networks with local information only. However, the absence of global information impedes the creation of efficient repairs. Motivated by this limitation, we propose a concept of community-level repair, which leverages the community structure of the network during the repair process. Moreover, we devise a general framework of network repair, with in total six instances. Evaluations on real-world and random networks show the effectiveness and efficiency of the community-level repair approaches, compared to local and random repairs. Our study contributes to a better understanding of repair processes, and reveals that exploitation of the community structure improves the repair process on a disrupted network significantly.

Introduction. – Modeling real-world systems as complex networks helps to assess and understand the resilience of the systems [1–3]. Complex network theory has been applied in many areas, including biological [4,5], economic [6,7], social [8–10], technological [11,12] and traffic networks [13,14]. Research has revealed network percolation processes under some conditions, which shows the vulnerability of networks [15–17]. One example of network vulnerability is the power blackout in northern and eastern India on July 30th, 2012 [18]. The blackout only resulted from the failure of circuit breakers on the 400KV Bina-Gwalior line, while this line was fed into one power station, and power failures cascaded through the grid, which affected over 300 million people, about 25% population of India, and it took 15 hours to restore 80% of service. Recent work on network percolation has successfully identified nodes in the network which are critical for the network, so-called influencers. A network attacked by decreasing (collective) influence breaks down faster than degree/random attacks [19]. Given such strong attacking strategies, it is of paramount importance to develop new repair strategies to recover networks quickly.

One view of network repair is to add new nodes or new links to the damaged network for functionality recovery [20–23]. This is the approach followed in our study. Nevertheless, in the literature other views have been proposed. Another network repair concept is to activate damaged nodes and links, aiming to reconnect components for restoring structural and functional features [24–26]. However, to activate failed nodes and links may be impractical in some cases, since the process to recover from damage could be very time-consuming. In addition to these two definitions of network repair, some researchers believe that redundant links are responsible for the connectivity of the networks, which means that the links for repair exist in advance [27,28] and other researchers study the protection of influential nodes from damage [29].

Existing research on adding new links during a repair process can be generally classified into two categories: 1)

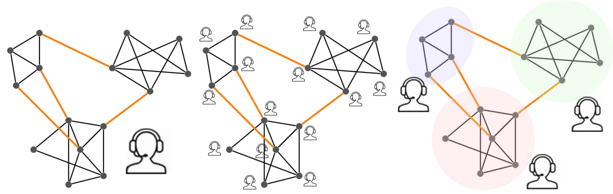
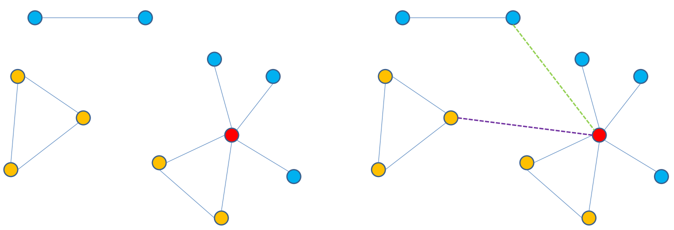


Fig. 1: Examples of global repair (left), local repair (center) and community repair (right).



(a) A damaged network with eleven nodes.

(b) Two possible new links under different strategies.

Fig. 2: Two kinds of possible new links under two node-level repair strategies.

local repair with very little information and 2) global repair, which needs more information and repair time. Both cases are visualized as shown in Fig.1. In global repair, the repair process is performed by an operator, and this operator possesses information of the whole network. Since the operator can access the whole network, he can develop an informed, global strategy for the network repair. Developing such a repair strategy needs a considerable amount of time, particularly on large networks. On the other hand, in recently proposed local repair strategies, each node only possesses information of itself, and the repair process is performed by each node spontaneously. The high computation efficiency of local repair makes it appealing for practical applications. The spontaneity of local repair, however, makes it difficult to control the repair quality and cost.

In this paper, we devise a novel repair strategy based on the concepts of communities [30, 31], as visualized in Fig.1 (center), in order to repair a network which was attacked by disruption of influencers (identified by collective influence). This strategy can be understood as an extension of local repair to greater sub networks. During the community-level repair process, the community structure of the network is leveraged to repair the network faster and cheaper than using only local information. Here, each community is controlled by an operator, communication between the operators is allowed and then new links are only generated between communities. The average efficiency [32] is used as the evaluation metric to compare local repair with community-level repair. The definition of the average efficiency is as follows:

$$E(G) = \frac{1}{N(N-1)} \sum_{i \neq j \in G} \frac{1}{d_{ij}} \tag{1}$$

where $E(G)$ is the average efficiency of the network G, N is the number of the nodes in the network, i and j are two nodes of the network, and d_{ij} stands for the shortest path length between node i and j.

Node-level repair strategies. – Since current local strategies focus on restoring functionality efficiently, the operations are performed on node level, such that, in general, no information of other nodes is needed. Three baseline node-level strategies are introduced here, in order to compare them to community-based repair below.

Random repair (Random). The most efficient repair strategy, in terms of computational resources required, is random repair: New links are generated randomly and no extra information is needed. Although the non-deterministic decisions cannot guarantee the effectiveness of the repair, this randomness makes it quite robust against a second attack, since the attackers cannot predict the new influencers after the repair.

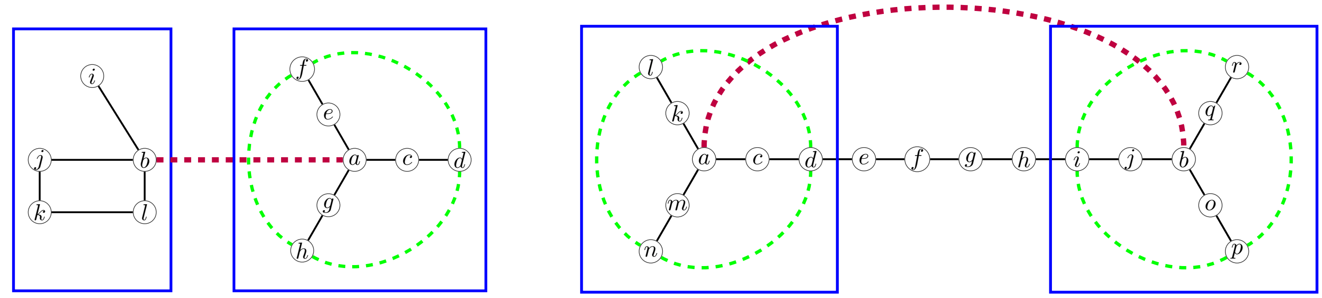
Node-level hub to hub repair (NHH). In order to improve the effectiveness of random repair, more important nodes could have larger probabilities to generate new links. There are many network metrics to evaluate the importance of nodes, but since degree is the most efficient to compute, it is used in our evaluation experiments. Each node of a new generated link is selected with the probability in proportion to the square of its degree, which can enlarge the difference between hub and non-hub nodes. The probability setting is chosen according to experiments and a power value of 2 works well in most cases.

Node-level hub to non-hub repair (NHN). In this strategy, new links between hub nodes and non-hub nodes have larger probabilities to be added. A hub to non-hub link connects the center and the periphery of the network, which would benefit the efficiency of the network, such as the green dashed links in Fig.2 (b).

Targeted community-level repair strategies. – Node is the smallest element in networks, including component and community. The number of candidate new links between nodes is, in general, $O(N^2)$ for sparse networks, where N is the number of nodes. Selecting the most effective link among these huge number of links is computationally complex, which results in many repair strategies with a trade-off between repair quality and response time.

In the targeted community repair strategies, each new link is generated only between the communities in the first two largest components, since nodes in the same community are densely connected already, by definition. For each community, only one node is chosen to represent the community to generate new links. Candidate links are selected from all the possible links between these representatives. From all candidates, we rank them according to the contribution they make to the average efficiency of the whole network, and the most effective one is generated, which can be classified as greedy ranking strategy. For each new link, all the operations above are iterated once, and after each generation, the community structure is updated. According to different rules for the candidate generation, three different strategies, whose results are deterministic and more effective, are proposed here:

Targeted community-level hub to hub repair (THH). In this strategy, the nodes with the largest degree represent the communities to generate the candidates.



(a) Average efficiency as the number of added links increases.

(b) The size of the giant component as the number of added links increases.

Fig. 3: Two kinds of new links in targeted community-level repair. Nodes in the same blue rectangle belong to the same community. The green circle is a ball of size=2, while the shortest path length of (a, b) in (b) is 9. Purple links are the candidates.

Targeted community-level hub to non-hub repair (THN). In this strategy, the candidates are generated between a node with the largest degree from a community and a node with the smallest degree from another community. It should be noted that the number of the candidates is nearly twice as that of the strategy above, so it needs more running time.

Targeted community-level hub to hub neighbor repair (THHN). In order to increase the robustness of the strategies, we also devise a non-deterministic strategy. For each community, we randomly select one neighbor of the hub node to represent the community, because this selection can confuse the attackers and neighbors of the same node would not cause huge difference to the effectiveness of the repair. New links are only generated between these neighbors.

As introduced above, for the greedy ranking process, the most effective link in terms of the average efficiency is generated. However, to compute the contribution of each link to the whole network average efficiency is quite complex, which is $O(N^2)$. Therefore, a much more efficient evaluation method for ranking candidate links is necessary. Since the definition of average efficiency is to calculate the mean of all the reciprocals of shortest path lengths, the mean inequality is used to estimate the average efficiency delta. The mean inequality is that between harmonic mean and arithmetic mean:

$$\frac{\sum_{i=1}^{n} x_i}{n} \geq \frac{n}{\sum_{i=1}^{n} \frac{1}{x_i}} \tag{2}$$

When the inequality is applied to the average efficiency, the lower bound of the average efficiency is obtained:

$$\frac{1}{N(N-1)} \sum_{i \neq j \in G} \frac{1}{d_{ij}} \geq N(N-1) \frac{1}{\sum_{i \neq j \in G} d_{ij}} \tag{3}$$

Since N is a constant value for a fixed network (the number of nodes in the network), the sum of the shortest path length of each node pair in the network can be used to evaluate the average efficiency delta. Therefore, we estimate the average efficiency delta by the sum of the shortest path lengths delta. Nevertheless, the traditional concept of the shortest path length is defined between two connected nodes, so it is necessary to define the path length between two disconnected nodes. In the definition of the average efficiency, if node i and j are disconnected, $d_{ij} = +\infty$, but this value of d_{ij} could not be used in the estimation process of the average efficiency delta. A considerably large value of d_{ij} always ranks the links between components in the first place, which is not the goal of our evaluation metric. In this paper, we define this path length to be the number of nodes in the network. It should be noted that a larger value of the path length between two disconnected nodes leads to more links between components. In addition, for each candidates, the time of calculating the average efficiency delta is $O(N^2)$, where N is the number of nodes, but this is computationally complex for quick repair response. To simplify the process to evaluate the effectiveness of all candidates, the closeness centrality is used here, with which to assess all candidate for one new link only needs time of $O(N^2)$. The definition of closeness centrality is:

$$c_i = \frac{N}{\sum_{j \in G} d_{ij}} \tag{4}$$

The candidate links are classified into two groups to be evaluated as follows.

For a new link (a, b) between two components as shown in Fig. 3 (a), the number of nodes in a's component is N_a and the number of nodes in b's component is N_b. c_a and c_b represent the closeness centrality of node a and node b respectively. The closeness centrality here is only computed based on the component of the node. Before the repair, all node pairs from different components are disconnected, such as node c and i. After the link (a, b) is

Table 1: Basic statistics for the eight networks used in our study. ASPL stands for average shortest path length. All networks are available for downloading at https://networkdata.ics.uci.edu/index.php or http://vlado.fmf.uni-lj.si/pub/networks/data/. ER means Erdös-Renyi network, BA is Barabasi-Albert network, WS stands for WattsStrogatz network and RG represents regular network. For random networks, different numbers represent model networks with different parameter settings.

Real-world networks					Random networks				
Network	|Nodes|	|Links|	Avg. degree	ASPL	Network	|Nodes|	|Links|	Avg. degree	ASPL
Epa	4253	8897	4.184	4.500	ER1	494	1233	4.992	4.067
Kohonen	2757	9804	7.112	3.185	ER2	500	1882	7.528	3.302
odlis	1347	1346	1.999	5.948	BA1	500	996	3.984	3.906
polblogs	1222	16717	27.360	2.738	BA2	500	1491	5.964	3.178
power	4941	6594	2.669	18.989	WS1	500	1000	4	5.473
SciMet	1421	6506	9.157	3.323	WS2	500	999	3.996	5.157
USAir97	332	2126	12.807	2.738	RG1	500	1999	7.996	3.256
Yeast	2225	7050	6.337	4.378	RG2	500	2499	9.996	2.950

generated, the shortest path lengths between nodes from the same component would not change, such as the shortest path length between nodes i and l, so all delta in terms of average efficiency comes from the path lengths between the nodes from different components, e.g. the shortest path length between node c and$i$ becomes 3 from that of disconnected nodes. Before the generation of a new link, the path length between nodes from two components is $Distance_{ori} = D_{uc} * N_a * N_b$, where D_{uc} means the path length of two disconnected nodes. In the paper, D_{uc} is set to be N, the number of nodes in the network. After the addition of the new link between components, all shortest paths between nodes from these two components should go through this new link. For instance, in Fig. 3 (a), after the generation of link (a, b), the shortest paths between nodes from the two components all go through link (a, b). Therefore, the sum of the shortest path lengths between each node in $component(a)$ and node b is $\sum_{x \in component(a)} (d_{ax} + 1)$. Assuming k_{bm} is the number of nodes at a distance of m from b in $component(b)$, the sum of the shortest path lengths between each node in $component(a)$ and these k_{bm} nodes is $k_{bm} * \sum_{x \in component(a)} (d_{ax} + k_{bm})$. Naturally, we have $1 + k_{b1} + \cdots + k_{b\infty} = N_b$ and $1 + 2 * k_{b1} + \cdots + m * k_{bm} + \cdots + \infty * k_{b\infty} = \sum_{x \in component(b)} (d_{bx} + 1)$. According to the formulations above, we finally obtain the sum delta of the shortest path lengths ($Delta_{betw}$) from this new link (a, b) between two components is:

$$\begin{aligned} Delta_{betw} = D_{uc} * N_a * N_b - \sum_{x \in component(a)} (d_{ax} + 1) \\ * N_b - \sum_{x \in component(b)} (d_{bx} + 1) * N_a + N_a * N_b \end{aligned} \tag{5}$$

According to the definition of the closeness centrality, the sum delta of the shortest path lengths ($Delta_{betw}$) could be computed as:

$$\begin{aligned} Delta_{betw} =& D_{uc} * N_a * N_b - \frac{1}{c_a} * (N_a - 1) * N_b \\ & - \frac{1}{c_b} * (N_b - 1) * N_a - N_a * N_b \end{aligned} \tag{6}$$

Through this formulation, the accurate delta of shortest path lengths is obtained.

For a new link (a, b) inside an existing component as shown in Fig. 3 (b), to compute the accurate delta with this link is very computationally complex, so an approximation is necessary to solve large-scale network problems. Similar to links between components, we want to obtain the path lengths delta after the generation of a new link. Assuming the shortest path length between node a and b is q_{ab} and k_{am} means the number of nodes at a distance of m from node a, the number of the shortest path lengths through this new link is at least $(1+k_{a1}+\cdots+k_{a\lfloor \frac{q_{ab}}{4} \rfloor})*(1+k_{b1}+\cdots+k_{b\lfloor \frac{q_{ab}}{4} \rfloor})$. In Figure 3 (b), for example, the shortest path lengths between nodes from different green dashed circles must pass through the new link (a, b), such as the shortest path between node k and q ($\{k \to a \to b \to q\}$). For each changed shortest path length, the distance delta of the central node pair (a, b) is used to represent each shortest path length delta, which is $q_{ab} - 1$. Therefore, we have the distance delta approximation ($Delta_{ins}$) for a new link inside a component:

$$\begin{aligned} Delta_{ins} =& (q_{ab} - 1) * (1 + k_{a1} + \cdots + k_{a\lfloor \frac{q_{ab}}{4} \rfloor}) * \\ & (1 + k_{b1} + \cdots + k_{b\lfloor \frac{q_{ab}}{4} \rfloor}) \end{aligned} \tag{7}$$

It should be noted that this formulation is an approximation of the shortest path lengths delta, but not an accurate value.

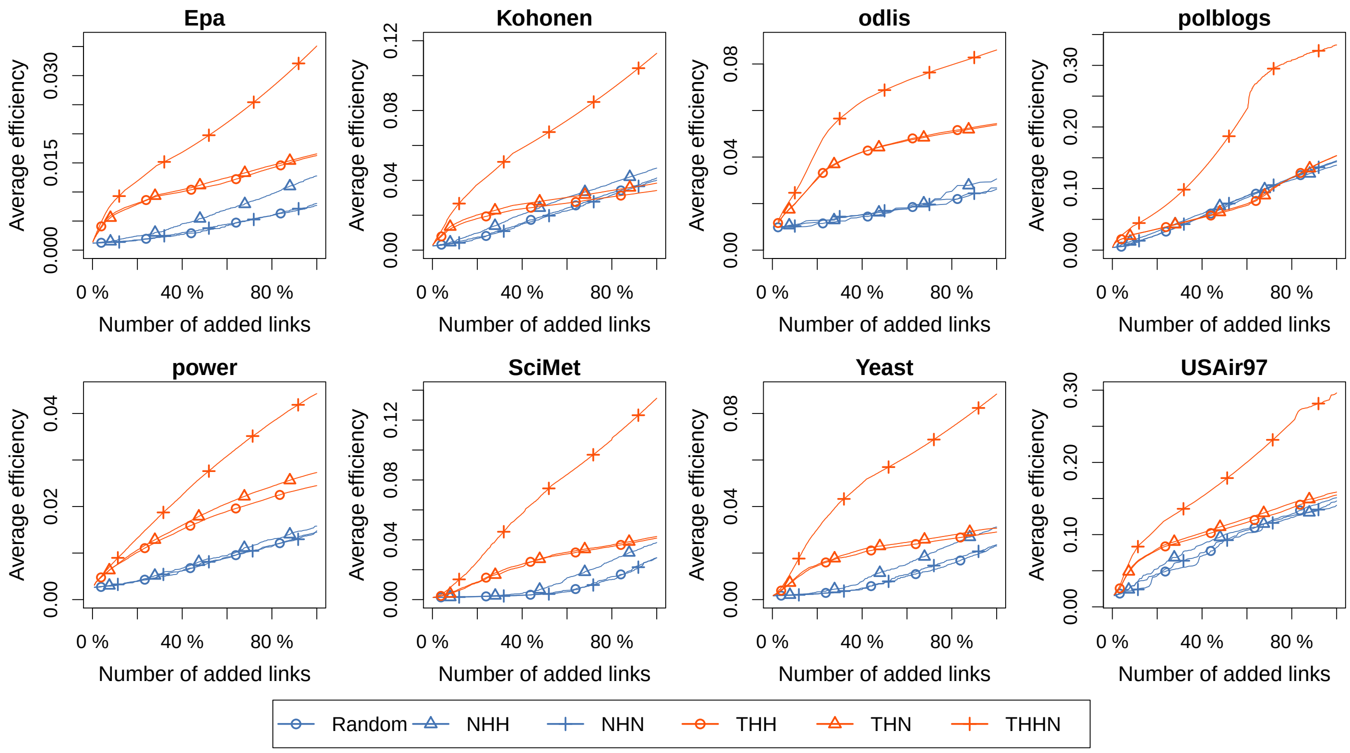


Fig. 4: The average efficiency of six repair strategies on eight real-world networks, as the number of added links increases, when the size of the giant component of each network is attacked to be less than 10%. The upper bound of the number of added links is 10% of that to make the repaired network be of the same density as the original network.

All candidates are ranked according to the distance delta introduced above, and for each generation of a new link, the most effective one is added to the network.

After the descriptions of the community-level repair strategy, the time complexity of generating one new link should be $O(NC^2 + N^2/C)$, where N is the number of the nodes in the network and C represents the community number in the largest two components of the network.

Evaluation results. – We evaluate all the six repair strategies on both eight real-world networks and four types of random networks. Since some of the repair strategies are non-deterministic, the results shown below are the medians of the data from experiments for 100 times. Table 1 shows the basic statistics for all networks. In order to compare the effectiveness of these strategies to damaged networks, we attack each network by the collective influence attacking strategy until the size of the giant component is less than 10% of the original network. Afterwards, we add 0 to 10% of the number of links, which are required for the repaired network to have the same density as the original network. The intuition is that the repair should try to restore the properties of the network and too many new links might nor be practical for real-world cases. The community detection method in this section is Louvain method [33], which is very efficient with an estimated computational complexity of $O(NlogN)$. Although it is well-established that random networks do not have any community structure, the nodes in random networks could be classified into different modules through Louvain method, and the repair process on the random networks is performed based on these modules.

Real-world networks. In Fig. 4, the average efficiency for eight real-world networks are shown, as the number of added links increases. Among the two concepts of repair strategy, so-called node-level repair and targeted community-level repair, the targeted community-level repair is the most effective in terms of average efficiency. Regarding the average efficiency in Fig. 4, the targeted community-level hub to hub neighbor repair behaves much better than the other two targeted repair strategies by nearly 40%, while the effectiveness of the other two methods is similar, among which targeted community-level hub to non-hub repair is a bit more effective than the other one. As for the node-level repair strategies, the node-level hub to hub repair strategy obtains a bit better effectiveness on these networks.

Although the evaluation metric in this paper is average efficiency, the targeted community-level strategies also performs well in terms of the size of the giant component, since new links between components are often important for the increase on average efficiency because this kind of links build new paths between disconnected node pairs. Of course, the targeted community-level would not always connect nodes from different components, but compare all

Table 2: Running time of one new link generation of all strategies on eight real-world networks. The unit of all numbers is seconds.

Strategies	Epa	Kohonen	odlis	polblogs	power	SciMet	USAir97	Yeast
Random	0.003	0.002	0.001	0.001	0.001	0.001	0.001	0.001
NHH	0.120	0.038	0.009	0.004	0.098	0.010	0.001	0.019
NHN	0.229	0.083	0.018	0.008	0.197	0.019	0.001	0.033
THH	7.166	3.443	0.752	0.599	9.245	1.037	0.099	1.706
THN	7.516	3.581	0.773	0.642	10.795	1.033	0.111	1.800
THHN	6.881	4.290	0.783	0.485	9.773	1.008	0.075	1.724

candidates and select the most effective one.

Table2 shows the running time of one new link generation of all strategies on the eight real-world networks. Since the running time for each new link on a network of a fixed size are very close, only the running time of one new link is shown here. As seen from the table, the node-level repair strategies could repair networks in very short time, while the targeted community-level strategies take more time. However, in view of the size of the networks, since one new link on a network with more than 4000 nodes only needs less than 10 seconds, the targeted community-level strategies can work in real-world larger networks.

Random networks. Fig. 5 shows the effectiveness of six strategies on four different kinds of random networks, and for each kind of random networks, two different parameter settings are used to evaluate the effectiveness on networks with diverse properties. The effectiveness of the six strategies is similar to that in real-world networks. In terms of average efficiency, the targeted community-level hub to hub neighbor strategy behaves much better than all the other strategies in these random networks, particularly in BA networks. In BA1 network, the average efficiency of targeted community-level hub to hub neighbor strategy is nearly twice of the second best strategy and five times of node-level strategies. Following the best strategy is the other two targeted community-level strategies, among which targeted community-level hub to non-hub strategy is a bit better than the other two. The node-level hub to hub strategy is the best among the three node-level strategies. Although the node-level repair strategies behave pretty well sometimes, such as in ER1 and RG1 networks, the randomness of these strategies makes it confusing whether the results is effective or not.

Discussion and conclusion. – In this paper, we first revisited local repair strategies based on establishing new connections between nodes, and then extended this concept to community-level repair, where new links are only generated between communities, in order to achieve a trade-off between the computational complexity and repair quality. Regarding the community-level repair, three targeted strategies were derived. The effectiveness of these strategies was evaluated both on real-world and random networks. The community-level repair strategies showed a good performance on all of the networks in terms of average efficiency. As for the community-level repair, the targeted community-level hub to hub neighbor strategy behaved much better than other two, particularly for relatively more added links. Compared to the community-level strategies, node-level strategies do not repair the network effectively. In conclusion, community-level repair is better than node-level repair for defending against CI, and among the community-level repair strategies, the targeted community-level hub to hub neighbor strategy is the most effective, as a trade-off between quality and efficiency. This study contributes a novel concept, community-structure-based repair, for resilience of real-world systems.

Future work can focus on the improvement of the efficiency of targeted community-level strategies and the influence of different community detection methods on the repair effectiveness. Our initial experiments indicate that the type of the community methods does not significantly influence the effectiveness. Moreover, for current targeted community-level strategies, after large communities are connected, the effectiveness of following new links generated by THH and THN decreases, so it is necessary to provide more limitations on the repair, where analysis of the community status is helpful for better quality.

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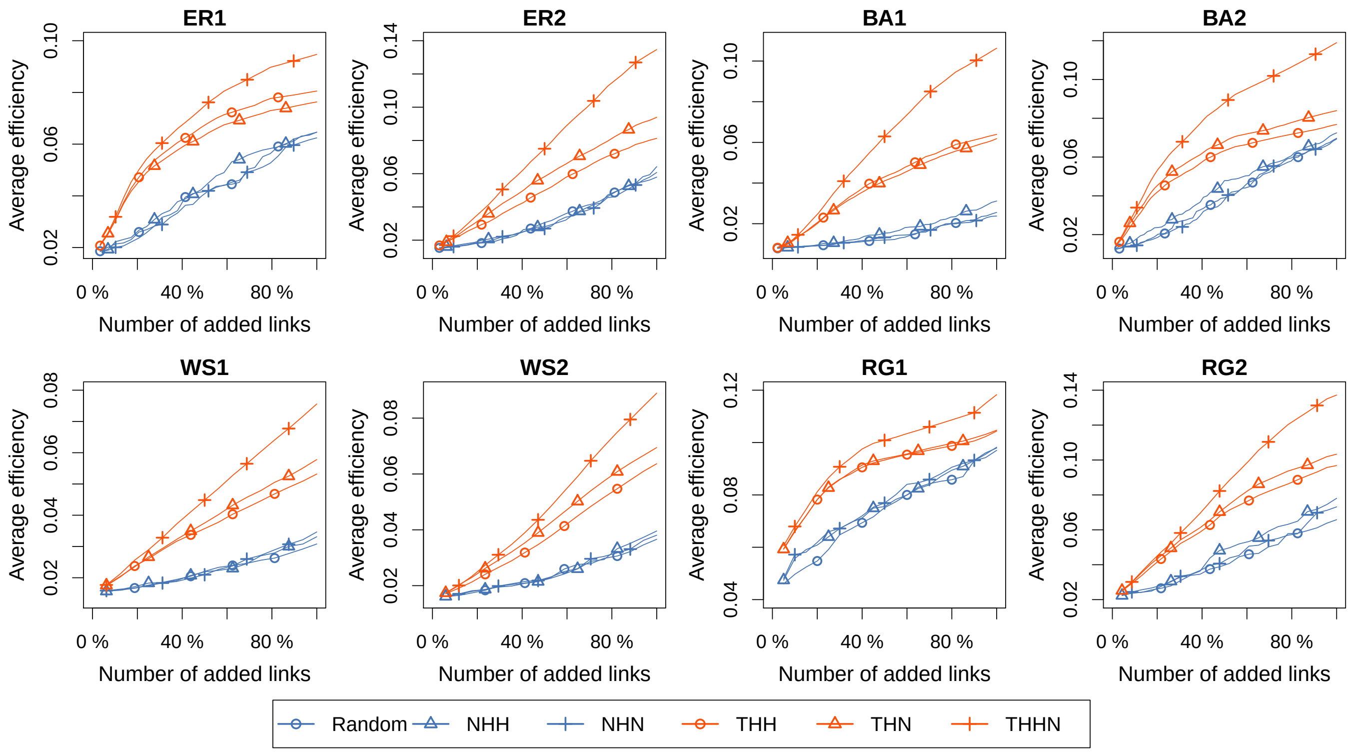

Fig. 5: The average efficiency of six repair strategies on eight random networks, as the number of added links increases, when the size of the giant component of each network is attacked to be less than 10%. The upper bound of the number of added links is 10% of that to make the repaired network be of the same density as the original network.

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